Review session: Mon Apr 13, 2-4pm. (?)
(poll on Piazza) maybe.

Remember: Teaching Evals.

Today: Cardinality. (Review).

infinite sets:

- countably infinite: $\mathbb{N}$, $\mathbb{Q}$ ...
- continuum: $\mathbb{R}$, $\mathcal{P}(\mathbb{N})$
- "unnaturally large": $\mathcal{P}(\mathbb{R})$

Worksheet 20: ① Want to prove: $A_1, \dots, A_n$ countable then $A_1 \times \dots \times A_n$ is countable.

- if all are finite, then the product is finite, $|A_1 \times \dots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$.
- if at least one is infinite, then the product is infinite, we need to prove it is countable.

Proof by induction: (if all are countably infinite, then product is countably infinite)

base: $n = 1$. $|A_1|$.
✓ nothing to prove.

induction step: Assumption: $A_1, \dots, A_k$ - count. inf.
$\Rightarrow A_1 \times \dots \times A_k$ is count. inf.

Need to prove: A_{k+1} is also countably inf.
$\Rightarrow A_1 \times \dots \times A_k \times A_{k+1}$
is also count. inf.

Lemma: A, B – count. inf. $\Rightarrow A \times B$ is countably infinite

(we proved: $\mathbb{N} \times \mathbb{N}$ is countably infinite

$f: \mathbb{N} \to A$ – bijective

$g: \mathbb{N} \to B$ – bijective

Then $h: \mathbb{N} \times \mathbb{N} \to A \times B$ – bijective.

$h(m,n) = (f(m), g(n))$)

Then we take $A = A_1 \times \ldots \times A_k$ ← count. inf. by the ind. assump.

$B = A_{k+1}$.

By Lemma, $A \times B$ is countably infinite, which completes pf of induction step.

Last thing: what if some of our sets are finite, and some are countably infinite?

lemma 1: "we can permute the factors":

i.e. $|A \times B| = |B \times A|$.

Pf: Need $f: A \times B \to B \times A$ bijective.

Define $f(a,b) = (b,a)$

easy exer: it is bijective.

This lemma allows us to collect the factors:

$$A_1 \times \ldots \times A_n = \underbrace{(\underbrace{A_{i_1} \times \ldots \times A_{i_n}}_{\text{finite ones}})}_{\text{finite, call it } B} \times (\underbrace{A_{j_1} \times \ldots \times A_{j_\ell}}_{\text{all infinite}})$$

(all infinite ↑ countably infinite, call it C)

Last lemma: if B is finite, C countably infinite, then

$B \times C$ is countably infinite.

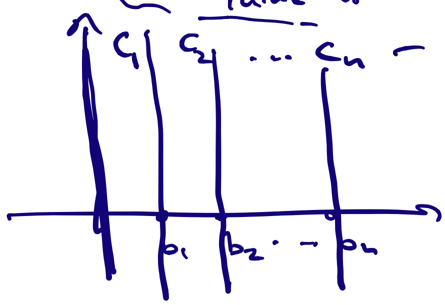
<u>Pf</u>: Suppose $n = |B|$. Let $B = \{b_1, \dots, b_n\}$.

Then $B \times C = C_1 \cup C_2 \cup \dots \cup C_n$

Let $C_i = \{(\underline{b_i}, c) \mid c \in C\}$

think of it as a copy of C

<u>ex</u>: $C = \mathbb{R}$



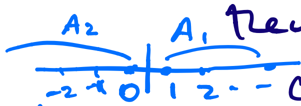
— each C_i is a copy of the line C

<u>Lemma</u>: A_1, A_2 - countably infinite $\Rightarrow A_1 \cup A_2$ is countably inf.

$f_1 : \mathbb{N} \to A_1$

$f_2 : \mathbb{N} \to A_2$

(same as pf that $|\mathbb{Z}| = |\mathbb{N}|$.)



Then by induction, $A_1 \cup \dots \cup A_n$ is countably infinite if $A_1, \dots, A_n$ is countably infinite.

$A_1 \cup A_2 \leftrightarrow \mathbb{Z}$. We proved $|\mathbb{Z}| = |\mathbb{N}|$.

This finishes the proof that $B \times C$ is countably infinite.

$f_1 : \mathbb{N} \to A_1$

$f_2 : \mathbb{N} \to A_2$

Define $f : \mathbb{Z} \to A_1 \cup A_2$

$$f(n) = \begin{cases} f_1(n) & \text{if } n > 0 \\ f_2(-(n+1)) & \text{if } n \le 0 \end{cases}$$

$f_2 : \mathbb{N} \to A_2$

transport it to $-\mathbb{N} \cup \{0\}$.

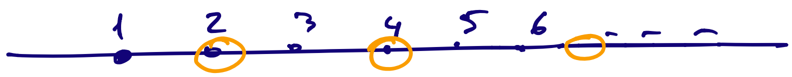
This function is surjective but may not be injective if $A_1 \cap A_2 \neq \emptyset$.

Then by Problem 3, $A_1 \cup A_2$ will be countably infinite..

Problems 2, 3: $f : A \to \mathbb{N}$ - injective.
Then A is finite or countably infinite.

(both are Theorems in the text)

Pf:



elements of $f(A)$.

Since f is injective, $f : A \to \underline{f(A)} \subset \mathbb{N}$ is bijective.

So we just need to prove that $f(A)$ is countable.

So we need to give a method for "numbering" the elements of $f(A)$.

Take the smallest one, call it b_1
the next one is b_2
...

this ends $\Leftrightarrow$ $f(A)$ is finite.

otherwise we have a way of labelling elements of $f(A)$ by natural numbers, so it is countably infinite.

/ Remark: to make this more precise:
axioms of natural numbers: "well-ordering axiom"
include "axiom of induction":

$\mathbb{N}$ is well-ordered set.

$$\boxed{\text{every non-empty subset of } \mathbb{N} \text{ has the smallest element}}$$

"obvious" about $\mathbb{N}$. axiom!

(note: $\mathbb{Z}$, or $\{a \in \mathbb{Q} : a > 0\}$ do not satisfy this. — It is a very strong statement!)

Our proof can be made more precise if we refer to this axiom:

Let b_1 = smallest element of $f(A)$ (exists by the axiom)

b_2 = smallest element of $f(A) - \{b_1\}$

...

b_n = smallest elt of $f(A) \setminus \{b_1, \dots, b_{n-1}\}$.

for every $n \in \mathbb{N}$.

So we established a bijection between $f(A)$ and $\mathbb{N}$, and f gives a bijection between A and $f(A)$, so we get

$$|A| = |f(A)| = |\mathbb{N}|.$$

Problem 3: $f: \mathbb{N} \to A$ surjective, then A is countable.

Trying to "number" the elements of A.

let $a_1 = f(1)$. let $a_2 = f(2)$ if $f(2) \neq f(1)$

if $f(2) = f(1)$,

let $a_2 = f(k)$, k = smallest natural number s.t.

$f(k) \neq f(1)$

What if $\{k \in \mathbb{N} : f(k) \neq f(1)\}$ is empty?

If this set is empty, $f(k) = f(1)$ $\forall k \in \mathbb{N}$

then $A = \{a_1\} = \{f(1)\}$ b/c f is surjective.

Then A is a set of one element.

So: we have a_1, and we defined a_2 or proved that $|A| = 1$.

Proceed as before:

$a_3 = f(k_3)$, where k_3 - smallest natural number s.t. $f(k_3) \neq a_1$ or a_2.

again, if such k_3 doesn't exist, then $|A| = 2$ (then $A = \{a_1, a_2\}$). b/c f is surjective.

. . .

If this process ends, A is finite.
if not, A is countably infinite (every element of A is labelled, b/c f is surjective).

Remark: in both problems, we "do not need all of $\mathbb{N}$" to label elements of A.

Problem 4:

$$A_1 = \{a_1^{(1)}, a_2^{(1)} \ldots a_n^{(1)} \ldots\}$$

$\vdots$

$$A_n = \{a_1^{(n)}, a_2^{(n)} \ldots, \ldots\}$$

helpful: group the finite ones together into one finite set, and prove:

1) if each $A_1, \ldots, A_n \ldots$ is countably infinite then the union is countably infinite.

2) A - countably infinite, B is finite, then $B \cup A$ is countably infinite. ← $\boxed{\text{exer}}$

Pf of (1): arrange them in a table, as before:

$$\begin{array}{llll} a_1^{(1)}, & a_2^{(1)}, & \cdots & \leftarrow \text{set } A_1 \\ a_1^{(2)}, & a_2^{(2)}, & \cdots & \leftarrow \text{set } A_2 \\ \vdots & & & \end{array}$$

Make a "snake path" visiting them all, as in the proof of $|\mathbb{N}\times\mathbb{N}| = |\mathbb{N}|$.

(or just say: let $f : \mathbb{N}\times\mathbb{N} \to \bigcup_{i=1}^{\infty} A_i$ be the function defined by
$$f(m,n) = a_n^{(m)}.$$
Then f is surjective (note: might not be injective)

So we have $f : \mathbb{N}\times\mathbb{N} \to \bigcup_{i=1}^{\infty} A_i$ – surjective.

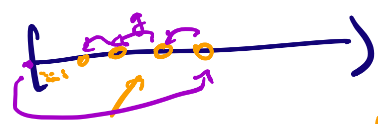
Since $|\mathbb{N}\times\mathbb{N}| = |\mathbb{N}|$, by the previous problem, $\left|\bigcup_{i=1}^{\infty} A_i\right| = |\mathbb{N}|$.

Problem 5 practice!

6 – easy ← do it!

Problems 7, 8. : 8 follows from 7.

#7: Trick:



make any countable infinite subset,

e.g. $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset (0,1)$

$(0,1) = A \cup B$, where $B = (0,1) \setminus A$.

$[0,1) = \{0\} \cup A \cup B$.

Now define $f : (0,1) \to [0,1)$ piecewise:

$$f(x) = \begin{cases} x, & \text{if } x \in B \\ g(x) & \text{if } x \in A \end{cases}$$

where g is a bijective function between A and $A \cup \{0\}$.

ex: $A = \{1/n \mid n \in \mathbb{N}\}$

define: $g\left(\frac{1}{n}\right) = \begin{cases} 1/n-1 & \text{if } n \neq 1 \\ 0 & \text{if } n = 1 \end{cases}$

You can always add / trow away a finite set from an infinite set, and the cardinality doesn't change.

also, can add/subtract a countable set ~~to~~ from uncountable set, and cardinality doesn't change.

9–11: last problems of HW 12.

sequences of 0s and 1s encode $\mathbb{R}$ ("binary"):